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Arithmetic–Geometric Mean determinantal identity

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ABSTRACT

In this paper, we give a generalization of a determinantal identity posed by Charles R. Johnson about minors of a Toeplitz matrix satisfying a specific matrix identity. These minors are those appear in the Dodgson's condensation formula.

1. Introduction

During the combinatorial workshop on combinatorics, linear algebra and graph theory in Tehran in 2003, Charles R. Johnson posed to the authors a problem related to evaluation of determinants using Dodgson's condensation formula [1,2]. To restate his problem in a more comprehensive way, we recall Dodgson's condensation formula.

Theorem 1.1 (Dodgson's condensation formula). *For any n by n matrix A, let $A_r[i, j]$ denote an r by r submatrix consisting of r contiguous rows and columns of A, starting with row i and column j. Then, we have*

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$$(\det A)\det A_{n-2}[2,2] = \det \begin{bmatrix} \det A_{n-1}[1,1] & \det A_{n-1}[1,2] \\ \det A_{n-1}[2,1] & \det A_{n-1}[2,2] \end{bmatrix}. \tag{1.1}$$

His problem is, as follows. Recall that a matrix A in the form $A = [a_{i-j}]$ is called a Toeplitz matrix, where $\{a_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers. Throughout the paper the superscript t stands for transposition of a matrix and also J_n is an all-one matrix of order n.

Problem 1.2 (Johnson's problem)**.** *Let A be a Toeplitz matrix of order n satisfying the matrix identity $A + A^t = 2J_n$. Then, we have the following determinantal identity:*

$$\frac{\det A_{n-1}[1,2] + \det A_{n-1}[2,1]}{2} = \det A_{n-1}[1,1]. \tag{1.2}$$

After working on the problem for few months, the authors realized that the condition of being "Toeplitz" can be relaxed and we can come up with the following interesting result.

Theorem 1.3 (Main theorem)**.** *Suppose A is a matrix and $A + A^t = \alpha J_n$, where α is a real number. Then, we have the following determinantal identity*

$$\underbrace{\sqrt{\det A_{n-1}[1,1]\det A_{n-1}[2,2]}}_{\text{Geometric Mean}} = \underbrace{\left|\frac{\det A_{n-1}[1,2] + \det A_{n-1}[2,1]}{2}\right|}_{\text{Arithmetic Mean}}. \tag{1.3}$$

2. Proof of the main theorem

Here, we prove a theorem and some corollaries that we use for proving the main theorem.

Theorem 2.1. *Let A be a matrix of order n and let β be a real number. Then*

$$\det(A + \beta J_n) = \det(A) - \beta \det \begin{bmatrix} 0 & 1 \cdots 1 \\ 1 & \\ \vdots & A \\ 1 & \end{bmatrix}.$$

Proof. If $u = [1, 1, \ldots, 1] \in \mathbb{R}^{1\times n}$, then

$$\det(A + \beta J_n) = \det(A + \beta u^t u) = \det \begin{bmatrix} 1 & u \\ -\beta u^t & A \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \cdots 1 \\ -\beta & \\ \vdots & A \\ -\beta & \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 \cdots 0 \\ -\beta & \\ \vdots & A \\ -\beta & \end{bmatrix} + \det \begin{bmatrix} 0 & 1 \cdots 1 \\ -\beta & \\ \vdots & A \\ -\beta & \end{bmatrix}$$

$$= \det(A) - \beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix}. \qquad \square$$

Corollary 2.2. *Let A be a matrix of even order and $A + A^t = \alpha J_n$, where α is a real number. Then for any real number β, we have*

$$\det(A + \beta J_n) = \det(A).$$

Proof. Considering Theorem 2.1, it is suffices to show that

$$\det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix} = 0.$$

We have

$$\det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \alpha J_n - A^t & \\ 1 & & & \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & -A^t & \\ 1 & & & \end{bmatrix}$$

$$= (-1)^n \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & A^t & \\ -1 & & & \end{bmatrix} = -\det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix}. \qquad \square$$

Remark 2.3. In Corollary 2.2, for $\alpha = 0$, A is a skew-symmetric matrix. In this case, a similar proof can be found in [3, Problem 1.2, p. 6].

Recall that a matrix A is a positive semidefinite matrix, if $x^tAx \geqslant 0$, for all vector x and also A is a negative semidefinite matrix, if $-A$ is a positive semidefinite matrix.

Remark 2.4. Considering $A + A^t = \alpha J_n$ and $x = [x_1, \ldots, x_n]^t$, we get

$$x^tAx = \frac{\alpha}{2}(x_1 + \cdots + x_n)^2.$$

If $\alpha \geqslant 0$, then A is a positive semidefinite matrix. Also, if $\alpha < 0$, then A is a negative semidefinite matrix.

Corollary 2.5. *Let A be a square matrix and $A + A^t = \alpha J_n$, where α is a real number. Put $B = A + \beta J_n$ where β is an arbitrary real number, we have*

(a) *If n is an even number*

$$\det A_{n-1}[2,1] - \det B_{n-1}[2,1] = \det A_{n-1}[1,2] - \det B_{n-1}[1,2].$$

(b) *If n is an odd number*

$$\det A_{n-1}[2,1] - \det B_{n-1}[2,1] = \det B_{n-1}[1,2] - \det A_{n-1}[1,2].$$

Proof. In Theorem 2.1, if we replace A with $A_{n-1}[2,1]$, then we get

$$\det(A_{n-1}[2,1] + \beta J_{n-1}) - \det A_{n-1}[2,1] = -\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A_{n-1}[2,1] & \\ 1 & & & \end{bmatrix}.$$

Since $B = A + \beta J_n$, we have $B_{n-1}[2,1] = A_{n-1}[2,1] + \beta J_{n-1}$, hence

$$\det B_{n-1}[2,1] - \det A_{n-1}[2,1] = -\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A_{n-1}[2,1] & \\ 1 & & & \end{bmatrix}. \tag{2.1}$$

Again using Theorem 2.1 and replacing A with $A_{n-1}[2,1]^t$, we have

$$\det(A_{n-1}[1,2]^t + \beta J_{n-1}) - \det A_{n-1}[1,2]^t = -\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A_{n-1}[1,2]^t & \\ 1 & & & \end{bmatrix}.$$

Since $B = A + \beta J_n$ and $A + A^t = \alpha J_n$, we obtain $B_{n-1}[1,2] = A_{n-1}[1,2] + \beta J_{n-1}$ and $A_{n-1}[1,2] + A_{n-1}[1,2]^t = \alpha J_n$, thus

$$\begin{aligned} \det B_{n-1}[1,2]^t - \det A_{n-1}[1,2] &= -\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \alpha J_{n-1} - A_{n-1}[1,2] & \\ 1 & & & \end{bmatrix} \\ &= -\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & -A_{n-1}[1,2] & \\ 1 & & & \end{bmatrix} \\ &= (-1)^{n-1}\beta \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A_{n-1}[1,2] & \\ 1 & & & \end{bmatrix}. \end{aligned} \tag{2.2}$$

Considering the relations (2.1) and (2.2) for even and odd n, we get the desired results. □

Now, we are in a position to prove the main result of this paper.

Proof. We distinguish between two cases:
Case I. Let n be an even number. Put $B = A - \frac{\alpha}{2}J_n$, where α is a real number. Since $A + A^t = \alpha J_n$, we get,

$$B^t = -B.$$

Therefore $B_{n-1}[1, 2]^t = -B_{n-1}[2, 1]$ and

$$\det B_{n-1}[1, 2] = -\det B_{n-1}[2, 1]. \tag{2.3}$$

Since $n-1$ is an odd number then $B_{n-1}[1, 1]$ is a skew-symmetric matrix of odd order, so

$$\det B_{n-1}[1, 1] = 0. \tag{2.4}$$

Using the fact that $B_{n-2}[2, 2] = A_{n-2}[2, 2] - \frac{\alpha}{2}J_{n-2}$ and $A + A^t = \alpha J_n$, and by Corollary 2.2 with $\beta = -\frac{\alpha}{2}$, we have

$$\det B_{n-2}[2, 2] = \det A_{n-2}[2, 2]. \tag{2.5}$$

$$\det B = \det A. \tag{2.6}$$

Hence by (2.5) and (2.6), we obtain

$$\det A_{n-2}[2, 2] \det A = \det B_{n-2}[2, 2] \det B. \tag{2.7}$$

Now, applying Dodgson's condensation formula on matrices A and B and using (2.7) and also (2.5) and (2.6), we get

$$\det A_{n-1}[1, 1] \det A_{n-1}[2, 2] - \det A_{n-1}[1, 2] \det A_{n-1}[2, 1] = \left(\det B_{n-1}[1, 2]\right)^2. \tag{2.8}$$

Since n is an even number, by Corollary 2.5 with $\beta = -\frac{\alpha}{2}$, we have

$$\det A_{n-1}[2, 1] - \det A_{n-1}[1, 2] = 2 \det B_{n-1}[1, 2].$$

Squaring the both sides of above equality and considering the formula (2.7), yields

$$\begin{aligned}(\det A_{n-1}[2, 1])^2 + (\det A_{n-1}[1, 2])^2 - 2 \det A_{n-1}[2, 1] \det A_{n-1}[1, 2] \\ = 4 \det A_{n-1}[1, 1] \det A_{n-1}[2, 2] - 4 \det A_{n-1}[1, 2] \det A_{n-1}[2, 1],\end{aligned}$$

Which gives us formula (1.3).

Case II. Suppose n is an odd number, so $n - 1$ is an even number. By Corollary 2.2 with $\beta = -\frac{\alpha}{2}$, we obtain

$$\begin{cases} \det A_{n-1}[1, 1] = \det B_{n-1}[1, 1], \\ \det A_{n-1}[2, 2] = \det B_{n-1}[2, 2]. \end{cases} \tag{2.9}$$

Now Corollary 2.5 with $\beta = -\frac{\alpha}{2}$, implies

$$\frac{\det A_{n-1}[1, 2] + \det A_{n-1}[2, 1]}{2} = \frac{\det B_{n-1}[1, 2] + \det B_{n-1}[2, 1]}{2}. \tag{2.10}$$

Finally, considering (2.9) and (2.10) and the Case (I), we get the desired result. □

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